Main Outlines

- Magnetic Field Energy
- Energy calculations in a magnetically coupled circuits
- Example

Magnetic Field Energy

- When a current is being established in a circuit, work has to be done
- If the current is ***i*** at a given instant and its rate of change is given by ***di/dt*** then the power being supplied by the *external* source is given by:

$$P = V_L i = L i \frac{di}{dt}$$

- The energy supplied is given by $dU = Pdt$
- The total energy stored in the inductor is then

$$U = L \int_0^I i \, di = \frac{1}{2} L I^2$$

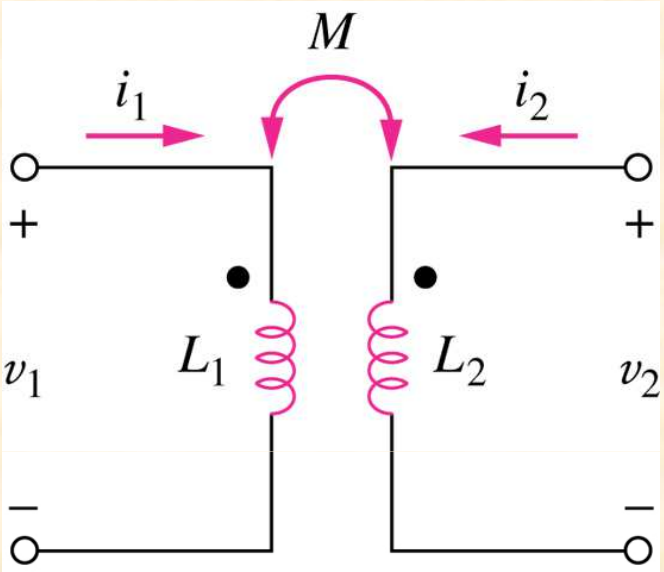
Energy in a Magnetically Coupled Circuits

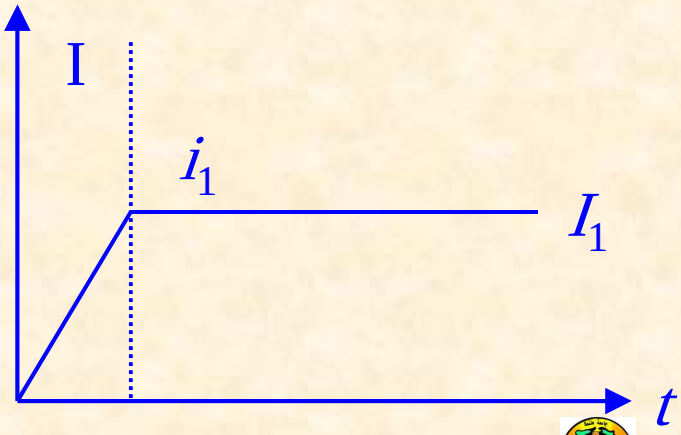
Step I :

$i_2 = 0$, i_1 increases from 0 to I_1

$$p_1(t) = i_1 v_1 = i_1 L_1 \frac{di_1}{dt}$$

$$\Rightarrow w_1 = \int p_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$





Energy in a Magnetically Coupled Circuits

Step II :

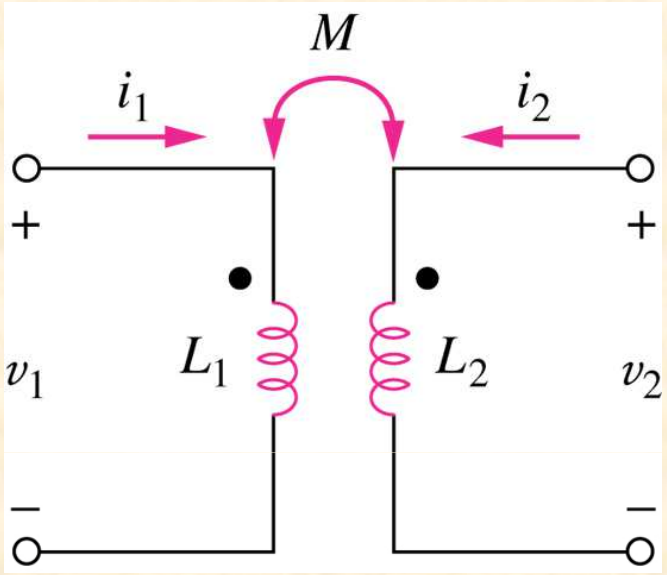
$i_1 = I_1$, i_2 increases from 0 to I_2

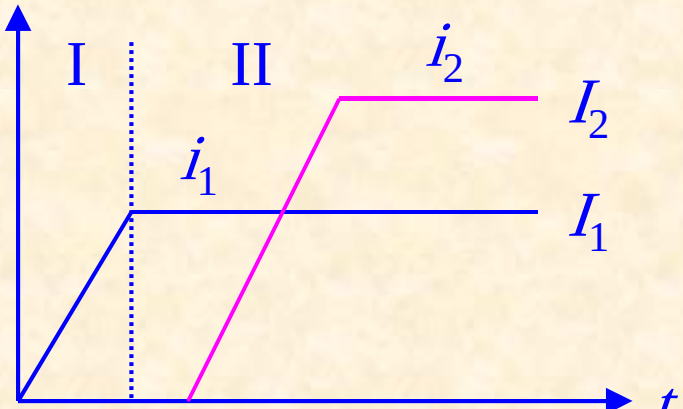
$$p_2(t) = i_1 v_1 + i_2 v_2 = I_1 M_{12} \frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

$$\Rightarrow w_2 = \int p_2 dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2$$

$$= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

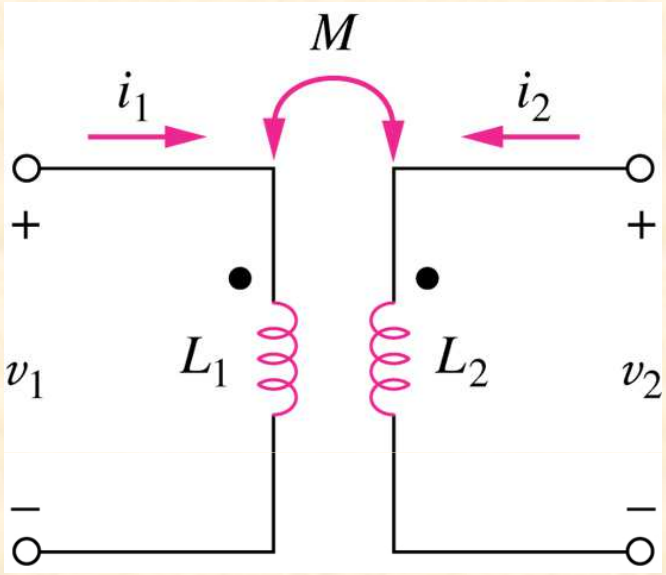
$$w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$





Energy in a Magnetically Coupled Circuits

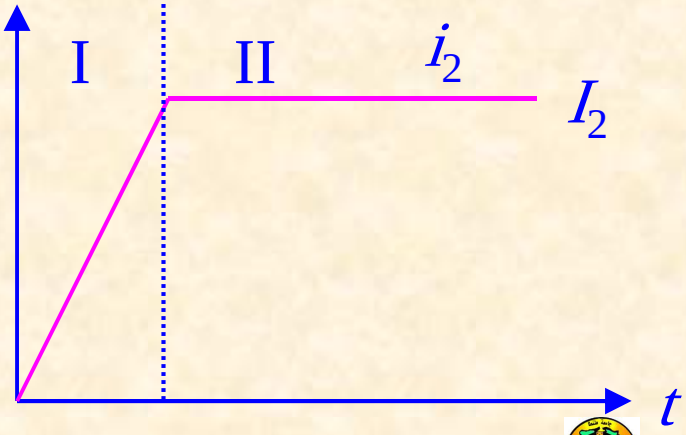
The analysis process can be changed as :



Step I :

$i_1 = 0$, i_2 increases from 0 to I_2

$$\Rightarrow w_1 = \frac{1}{2} L_2 I_2^2$$



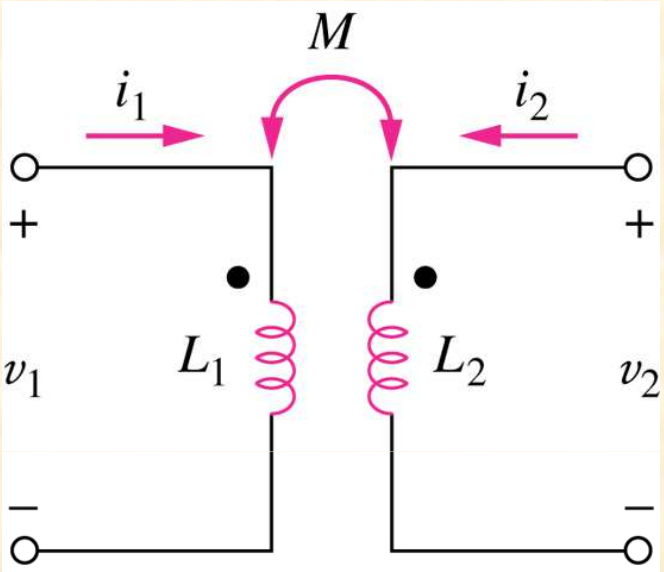
Energy in a Magnetically Coupled Circuits

Step II :

$i_2 = I_2$, i_1 increases from 0 to I_1

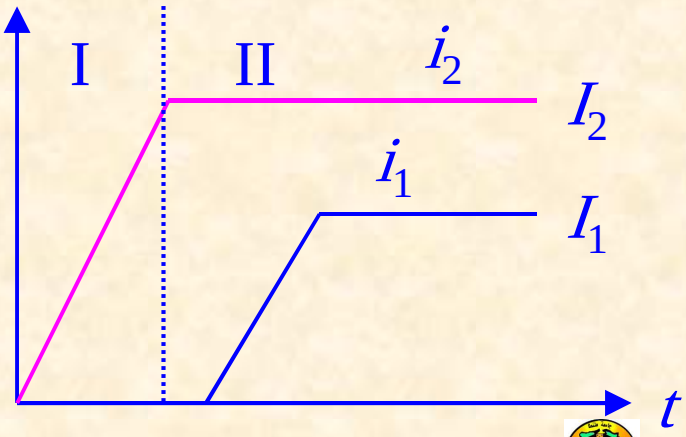
$$\Rightarrow w_2 = M_{21} I_1 I_2 + \frac{1}{2} L_1 I_1^2$$

$$w = w_1 + w_2 = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2$$



However, the total energy must equal to the former case

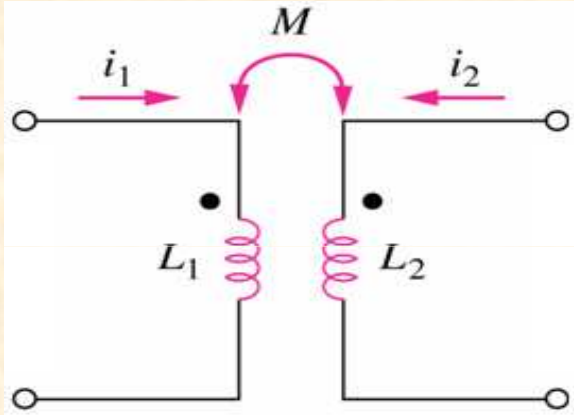
$$\Rightarrow M_{12} = M_{21} = M$$

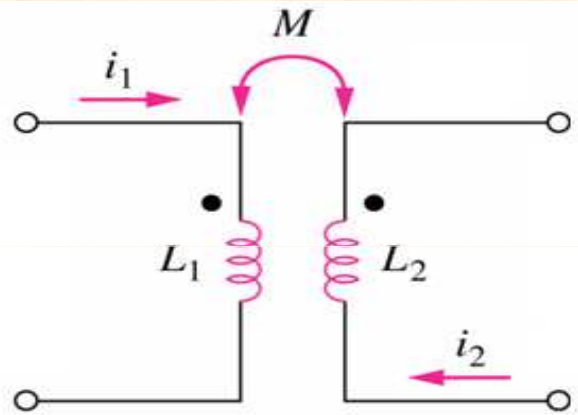


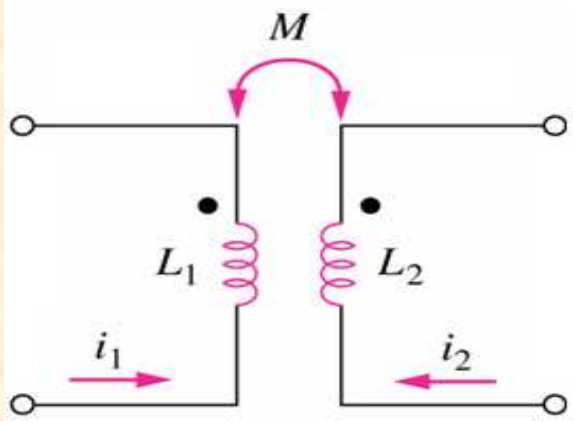
Energy in a Magnetically Coupled Circuits

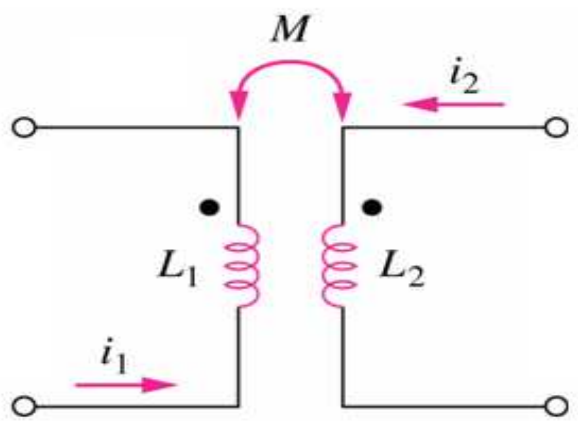
$$w = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2$$

$$w = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 - M i_1 i_2$$









Energy in a Magnetically Coupled Circuits

- The total energy w stored in a mutually coupled inductor is:

$$w = \frac{1}{2} L_1 i_1^{\,2} + \frac{1}{2} L_2 i_2^{\,2} \pm M i_1 i_2$$

✓ Positive sign is selected if both currents ENTER or LEAVE the dotted terminals

✓ Otherwise, use Negative sign

Energy in a Magnetically Coupled Circuits

- The instantaneous energy stored is given by:

$$w = \frac{1}{2} L_1 i_1^{\,2} + \frac{1}{2} L_2 i_2^{\,2} \pm M i_1 i_2$$

However, $w > 0$

Therefore, $$\frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 - M i_1 i_2 \geq 0$$

Energy in a Magnetically Coupled Circuits

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0$$

$$\frac{1}{2}\left(i_1\sqrt{L_1} - i_2\sqrt{L_2}\right)^2 + i_1 i_2\left(\sqrt{L_1 L_2} - M\right) \geq 0$$

$$\Rightarrow \sqrt{L_1 L_2} - M \geq 0$$

$$\Rightarrow \boxed{M \leq \sqrt{L_1 L_2}}$$

Energy in a Magnetically Coupled Circuits

- The coupling coefficient "*k*" is defined as:

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

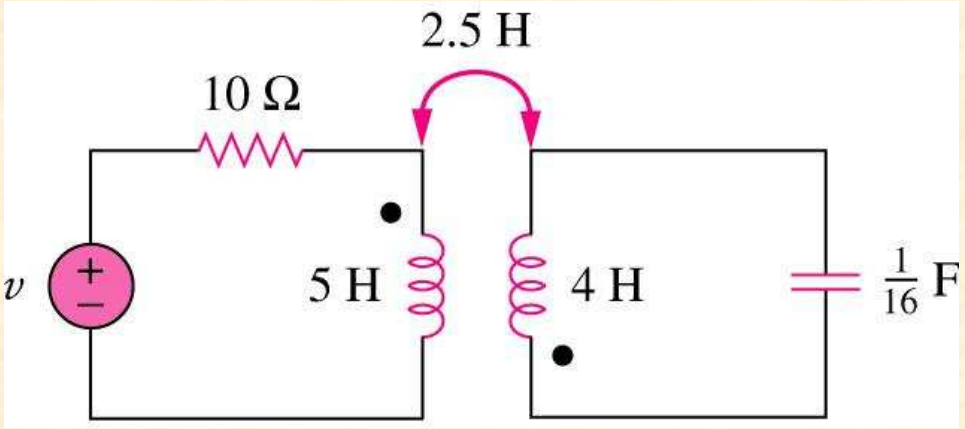
$$\text{or} \quad M = k\sqrt{L_1 L_2} \qquad (0 \le k \le 1)$$

- This ensures that:

Range of k : $0 \le k \le 1$

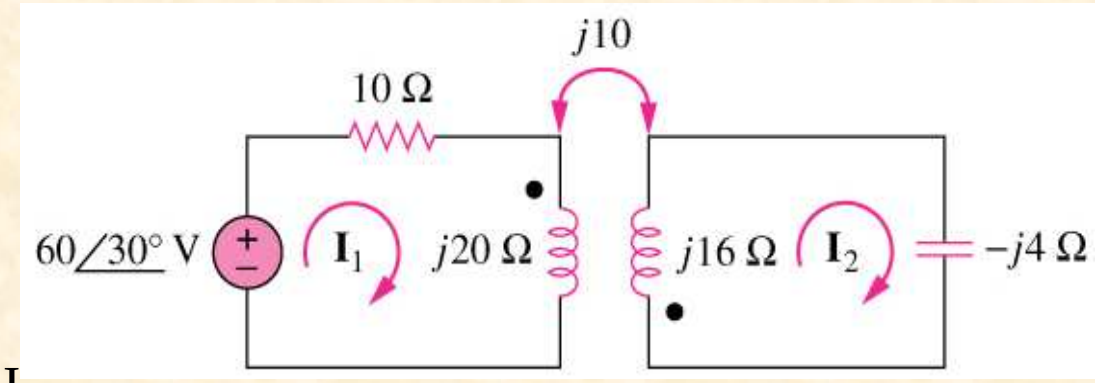
➢ Example

Find k and the energy stored in the coupled inductors at $t = 1$ s.



$$v = 60\cos(4t + 30^\circ)\ \text{V}$$

$$\omega = 4\ \text{rad/s}$$



$$\text{Sol}: \quad k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

For mesh 1,

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = \frac{60}{\sqrt{2}}\angle 30^\circ \quad (1a)$$

For mesh 2,

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0 \quad (1b)$$

$$\Rightarrow \begin{cases} \mathbf{I}_1 = \left(3.904/\sqrt{2}\right)\angle -19.4^\circ \\ \mathbf{I}_2 = \left(3.254/\sqrt{2}\right)\angle 160.6^\circ \end{cases}$$

$$\Rightarrow \begin{cases} i_1 = 3.904\cos(4t - 19.4^\circ) \\ i_2 = 3.254\cos(4t + 160.6^\circ) \end{cases}$$

$$\Rightarrow i_1(1) = 3.764, \quad i_2(1) = -3.137$$

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 = 84.62\ \text{J}$$